Cuntz-Pimsner Algebras and a potential L^p version.

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Abstract

The Fock space construction for Hilbert modules gives rise to an important class of C*-algebras, the so called Cuntz-Pimsner Algebras, which includes some well known examples of C*-algebras like the Cuntz algebras and crossed products by $\mathbb{Z}$. Back in 2012, Chris Phillips introduced an analog of the Cuntz algebras on L^p spaces and the theory of crossed products of L^p operator algebras. Recently, I've been working on a Fock space type construction that I think will produce a class of L^p operator algebras that hopefully will contain the L^p Cuntz algebras and the L^p version of crossed products by $\mathbb{Z}$.

I will start this talk with an explicit construction of the usual Cuntz algebras. Then, I will explain how to construct the Toeplitz algebra $\mathcal{T}_E$ and the Cuntz-Pimsner algebra $\mathcal{O}_E$ for a Hilbert A-module E. I will prove that the Cuntz algebras are a special case of this construction. Time permitting, I will say something about how a version of this could work in the L^p-setting, which tools I'll need to use and explain what are the most likely problems I will encounter down the road.

1 The Cuntz Algebras $\mathcal{O}_d$ and $\mathcal{E}_d$

Let $d \in \mathbb{Z}_{\geq 2}$ and $\mathcal{H}$ an infinite dimensional separable Hilbert space. Then, there are elements $s_1, s_2, \ldots, s_d \in \mathcal{L}(\mathcal{H})$ such that

$$s_j^* s_j = 1 \quad \text{and} \quad \sum_{j=1}^{d} s_j s_j^* = 1 \tag{1}$$

Let's show this for $d = 2$. Let $\{\xi_1, \xi_2, \ldots\}$ be an orthonormal basis for $\mathcal{H}$. Define $s_j : \mathcal{H} \to \mathcal{H}$ for $j = 1, 2$ by

$$s_1(\xi_n) = \xi_{2n} \quad \text{and} \quad s_2(\xi_n) = \xi_{2n-1} \qquad n \geq 1$$

One quickly checks that

$$s_1^*(\xi_n) = \begin{cases} \xi_{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad s_2^*(\xi_n) = \begin{cases} \xi_{(n+1)/2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

To check that the relations 1 are satisfied, we look first at $s_j^* s_j$ for $j = 1, 2$:

$$\begin{array}{ccccc} & s_1 & & s_1^* & \\ \xi_1 & \mapsto & \xi_2 & \mapsto & \xi_1 \\ \xi_2 & \mapsto & \xi_4 & \mapsto & \xi_2 \\ \xi_3 & \mapsto & \xi_6 & \mapsto & \xi_3 \\ \xi_4 & \mapsto & \xi_8 & \mapsto & \xi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \quad \text{and} \quad \begin{array}{ccccc} & s_2 & & s_2^* & \\ \xi_1 & \mapsto & \xi_1 & \mapsto & \xi_1 \\ \xi_2 & \mapsto & \xi_3 & \mapsto & \xi_2 \\ \xi_3 & \mapsto & \xi_5 & \mapsto & \xi_3 \\ \xi_4 & \mapsto & \xi_7 & \mapsto & \xi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

so indeed $s_j^* s_j = 1$. We now look at $s_j s_j^*$ for $j = 1, 2$:

$$\begin{array}{ccccc} & s_1^* & & s_1 & \\ \xi_1 & \mapsto & 0 & \mapsto & 0 \\ \xi_2 & \mapsto & \xi_1 & \mapsto & \xi_2 \\ \xi_3 & \mapsto & 0 & \mapsto & 0 \\ \xi_4 & \mapsto & \xi_2 & \mapsto & \xi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \quad \text{and} \quad \begin{array}{ccccc} & s_2^* & & s_2 & \\ \xi_1 & \mapsto & \xi_1 & \mapsto & \xi_1 \\ \xi_2 & \mapsto & 0 & \mapsto & 0 \\ \xi_3 & \mapsto & \xi_2 & \mapsto & \xi_3 \\ \xi_4 & \mapsto & 0 & \mapsto & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

from were we clearly see that $s_1 s_1^* + s_2 s_2^* = 1$.

Definition 1.1. We define $\mathcal{O}_d$, the Cuntz algebra of order $d \in \mathbb{Z}_{\geq 2}$, as the sub-C* algebra in $\mathcal{L}(\mathcal{H})$ generated by $s_1, \ldots, s_n$.

The construction of $\mathcal{O}_d$ is independent of the Hilbert space $\mathcal{H}$ and the choice of isometries s_j as long as the relations 1 are satisfied.

The algebra $\mathcal{O}_d$ is a simple, unital C*-algebra and has the following universal property: If A is a unital C*-algebra containing elements $a_1, \ldots, a_d$ such that

$$a_j^* a_j = 1 \quad \text{and} \quad \sum_{j=1}^{d} a_j a_j^* = 1,$$

then there is a unique $*$-homomorphism $\varphi : \mathcal{O}_d \to A$ such that $\varphi(s_j) = a_j$.

Definition 1.2. For $d \in \mathbb{Z}_{\geq 2}$, look at the generating isometries $s_1, \ldots, s_{d+1} \in \mathcal{O}_{d+1}$ and let $\mathcal{E}_d$ be the sub-C* algebra in $\mathcal{L}(\mathcal{H})$ generated by $s_1, \ldots, s_d$. That is, $\mathcal{E}_d$ is the universal unital C*-algebra generated by d isometries, whose orthogonal ranges do not add up to 1.

The Cuntz algebra $\mathcal{O}_d$ has elements satisfying the relations of $\mathcal{E}_d$, so by universality there is a surjective map $\mathcal{E}_d \to \mathcal{O}_s$. The kernel of this map is the ideal in $\mathcal{E}_d$ generated by $s_{d+1} s_{d+1}^* = 1 - \sum^d s_j s_j^*$, which we denote by $\mathcal{J}_d$. Then $\mathcal{E}_d / \mathcal{J}_d \cong \mathcal{O}_d$.

2 A brief review of Hilbert Modules

Definition 2.1. Let A be a C*-algebra and E a complex vector space which is also a right A-module. An **A-valued right inner product on E** is a map

$$\begin{array}{ccc} E \times E & \to & A \\ (\xi, \eta) & \mapsto & \langle \xi, \eta \rangle_A \end{array}$$

such that for any $\xi, \eta, \eta_1, \eta_2 \in E$, $a \in A$ and $\alpha \in \mathbb{C}$ we have

1. $\langle \xi, \eta_1 + \alpha \eta_2 \rangle_A = \langle \xi, \eta_1 \rangle_A + \alpha \langle \xi, \eta_2 \rangle_A$.
2. $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$.
3. $\langle \xi, \eta \rangle_A^* = \langle \eta, \xi \rangle_A$.
4. $\langle \xi, \xi \rangle_A \geq 0$ in A.
5. $\langle \xi, \xi \rangle_A = 0 \implies \xi = 0$.

Definition 2.2. Let A be a C*-algebra. A **Hilbert A-module** is a complex vector space E which is a right A-module with an A-valued right inner product and so that E is complete with the norm $\|\xi\| := \|\langle\xi,\xi\rangle_A\|^{1/2}$. We say that E is **full** if the two sided ideal $\operatorname{span}\langle E,E\rangle_A := \operatorname{span}\{\langle\xi,\eta\rangle_A : \xi,\eta \in E\}$ is dense in A.

Example 2.3. Let $\mathcal{H}$ be a Hilbert space with the physicists convention that the inner product is linear in the second variable. Then, $\mathcal{H}$ is clearly a full Hilbert $\mathbb{C}$-module.

Example 2.4. Any C^*-algebra A is clearly a full Hilbert A-module with inner product given by $(a,b) \mapsto a^*b$. More generally, A^n is also a full Hilbert A-module with the obvious "euclidean" inner product.

Example 2.5. The set of continuous sections of a vector bundle over a compact Hausdorff space X equipped with a Riemannian metric g is a Hilbert $C(X)$-module.

Example 2.6. If $(E_\lambda)_{\lambda\in\Lambda}$ is an arbitrary family of Hilbert A-modules, we can form their direct sum

$$\bigoplus_{\lambda\in\Lambda} E_\lambda := \left\{\xi = (\xi_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} E_\lambda : \sum_{\lambda\in\Lambda}\langle\xi_\lambda,\xi_\lambda\rangle \text{ converges in } A\right\}$$

which is a right A-module with coordinate-wise action and it becomes a Hilbert A-module when equipped with the well defined A-valued inner product

$$\langle\xi,\eta\rangle := \sum_{\lambda\in\Lambda}\langle\xi_\lambda,\eta_\lambda\rangle$$

A main difference between Hilbert modules and Hilbert spaces is that not every bounded linear map between Hilbert A-modules has an adjoint. We will only be interested in those maps that do have an adjoint.

Definition 2.7. Let E and F be a Hilbert A-modules. A map $t : E \to F$ is said to be **adjointable** if there is a map $t^* : F \to E$ such that for any $\xi \in E$, and $\eta \in F$

$$\langle t(\xi),\eta\rangle = \langle\xi, t^*(\eta)\rangle$$

The space of adjointable maps from E to F is denoted by $\mathcal{L}_A(E,F)$ and $\mathcal{L}_A(E) := \mathcal{L}_A(E,E)$.

It's almost immediate that adjointable maps between Hilbert modules are linear and bounded. A standard proof shows that $\mathcal{L}_A(E)$ is a C*-algebra when equipped with the operator norm. We will have special interest for a particular case of andjointable maps, those of "rank 1":

Definition 2.8. Let E and F be a Hilbert A-modules. For $\xi \in E$ and $\eta \in F$, we define a map $\theta_{\xi,\eta} : F \to E$ by

$$\theta_{\xi,\eta}(\zeta) := \xi\langle\eta,\zeta\rangle_A$$

One easily checks that $\theta_{\xi,\eta} \in \mathcal{L}_A(E,F)$, that $(\theta_{\xi,\eta})^* = \theta_{\eta,\xi} \in \mathcal{L}_A(F,E)$ and that $\|\theta_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$. This gives an analogous of the class of rank-one operators on Hilbert spaces. So, we define an analogous of the compact operators by letting

$$\mathcal{K}_A(E,F) := \overline{\operatorname{span}\{\theta_{\xi,\eta} : \xi \in E, \eta \in F\}}$$

It's also not hard to verify that $\mathcal{K}_A(E) := \mathcal{K}_A(E,E)$ is a closed two sided ideal in $\mathcal{L}_A(E)$, whence $\mathcal{K}(E)$ is also a C^*-algebra.

3 The Fock space construction

Definition 3.1. Let A be a C*-algebra. A C*-correspondence over A is a pair (E, φ) where E is a Hilbert right A-module and a $\varphi : A \to \mathcal{L}_A(E)$ is a $*$-homomorphism.

Remark 3.2. Even though it's not strictly necessary, from now on we will assume that the map φ of a C*-correspondence over A is injective and therefore isometric. This is done for simplicity.

Given (E, φ_E) and (F, φ_F) two C*-correspondence over A, we use the inner tensor product construction to produce $(E \otimes_{\varphi_F} F, \widetilde{\varphi_E})$, a new C*-correspondence over A. More precisely, $E \otimes_{\varphi_F} F$ is a Hilbert module such that the middle action is glued:

$$\xi a \otimes \eta - \xi \otimes \varphi_F(a)\eta = 0,$$

it's an A-module with right action satisfying

$$(\xi \otimes \eta)a = \xi \otimes (\eta a),$$

it has an A-valued right inner product such that

$$\langle \xi_1 \otimes \eta_1, \xi_1 \otimes \eta_2 \rangle = \langle \eta_1, \varphi_F(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle.$$

In fact, $E \otimes_{\varphi_F} F$ is the completion of the algebraic tensor product $E \odot_A F$ with respect to the norm induced by the above inner product. Finally, $\widetilde{\varphi_E} : A \to \mathcal{L}_A(E \otimes_{\varphi_F} F)$ is defined on elementary tensors by

$$\widetilde{\varphi_E}(a)(\xi \otimes \eta) = (\varphi_E(a)(\xi)) \otimes \eta$$

From now on we'll do an abuse of notation and drop the $\tilde{}$ on top of φ_F. In fact, any adjointable map acting on a Hilbert A module, also acts on $E \otimes -$ by only acting on the E portion.

Definition 3.3. Given (E, φ) a C*-correspondence over A, the Fock space of E is the Hilbert A-module given by

$$\mathcal{F}(E) := \bigoplus_{n \geq 0} E^{\otimes n},$$

where $E^{\otimes 0} := A$ and $E^{\otimes n} := \underbrace{E \otimes_\varphi \ldots \otimes_\varphi E}_{n \text{ times}}$.

An arbitrary element of $\mathcal{F}(E)$ is a tuple $(\kappa_n)_{n \geq 0}$ where each κ_n is an element of the nth degree tensor product of E.

For a fixed $\xi \in E$ and any $n \in \mathbb{Z}_{\geq 1}$ we have a *creation operator* $c_\xi : E^{\otimes n} \to E^{\otimes (n+1)}$ given by

$$c_\xi(\eta) := \xi \otimes \eta, \quad \forall\, \eta \in E^{\otimes n}$$

If $n = 0$ we set $c_\xi : A \to E$

$$c_\xi(a) := \xi a, \quad \forall\, \in A$$

Each c_ξ is an adjointable map where, if $n \in \mathbb{Z}_{\geq 1}$, $c_\xi^* : E^{\otimes (n+1)} \to: E^{\otimes n}$ is an *annihilation operator*, satisfying

$$c_\xi^*(\zeta \otimes \eta) = \varphi(\langle \xi, \zeta \rangle)\eta, \quad \forall\, \zeta \in E, \eta \in E^{\otimes n}$$

and $c_\xi^* : E \to A$ is simply

$$c_\xi^*(\zeta) = \langle \xi, \zeta \rangle, \quad \forall\, \zeta \in E,$$

Notice that c_ξ increases the degree by one, whereas c_ξ^* decreases the degree by one. Moreover, let $\xi, \zeta \in E$ and $n \geq 0$. We have the following important property for the map $c_\xi^* c_\zeta : E^{\otimes n} \to E^{\otimes n}$:

$$c_\xi^* c_\zeta = \varphi(\langle \xi, \zeta \rangle) \in \mathcal{L}_A(E^{\otimes n}),$$

and that also for the map $c_\xi c_\zeta^* : E^{\otimes(n+1)} \to E^{\otimes(n+1)}$, which satisfies

$$c_\xi c_\zeta^* = \theta_{\xi,\zeta} \in \mathcal{L}_A(E^{\otimes(n+1)}).$$

We abuse notation and consider the elements c_ξ as elements of $\mathcal{L}_A(\mathcal{F}(E))$ acting coordinate-wise:

$$c_\xi\big((\kappa_n)_{n\geq 0}\big) := \big(c_\xi(\kappa_n)\big)_{n\geq 0}$$

and

$$c_\xi^*\big((\kappa_n)_{n\geq 0}\big) := \big(c_\xi^*(\kappa_n)\big)_{n\geq 1}$$

4 The Toeplitz algebra $\mathcal{T}_E$ and $\mathcal{T}_{\mathbb{C}^d} \cong \mathcal{E}_d$

Definition 4.1. Let (E, φ) be a C^*-correspondence over A. We define $\mathcal{T}_E$, the Toeplitz algebra of E, as the C^*-subalgebra in $\mathcal{L}_A(\mathcal{F}(E))$ generated by the creation operators $\{c_\xi : \xi \in E\}$.

Remark 4.2. Notice that the definition of $\mathcal{T}_E$ does not change if instead of A we use $\overline{\text{span}}\langle E, E\rangle \subseteq A$. Thus, we might as well assume that E is a full Hilbert A-module. We then can identify A with it's image in $\mathcal{T}_E$ given by $a \mapsto \varphi(a)$. Indeed, it suffices to check that $\varphi(a) \in \mathcal{T}_E$ for any $a \in A$. By the fullness assumpltion, it suffices to check it when $a = \langle \xi, \zeta\rangle$, but we already know that $\varphi(\langle \xi, \zeta\rangle) \in c_\xi^* c_\zeta \in \mathcal{T}_E$.

The Toeplitz algebra $\mathcal{T}_E$ has the following universal property. Suppose B is another C^*-algebra and that there is a $*$-homomoprhism $\pi : A \to B$, a linear map $t : E \to B$ such that

1. $t(\xi)^* t(\zeta) = \pi(\langle \xi, \zeta\rangle)$ for $\xi, \zeta \in E$,
2. $\pi(a)t(\xi) = t(\varphi(a)\xi)$

Then there π has is a unique extension $\widehat{\pi} : \mathcal{T}_E \to B$ that sends c_ξ to $t(\xi)$.

Theorem 4.3. *Let $d \in \mathbb{Z}_{\geq 2}$ and regard $\mathbb{C}^d$ as a Hilbert $\mathbb{C}$-module. Let $\varphi : \mathbb{C} \to \mathcal{L}_{\mathbb{C}}(\mathbb{C}^d)$ by given by*

$$\varphi(z)(\zeta_1, \ldots, \zeta_d) := (z\zeta_1, \ldots, z\zeta_d)$$

Then $(\mathbb{C}^d, \varphi)$ is a C^* correspondence and $\mathcal{T}_{\mathbb{C}^d} \cong \mathcal{E}_d$.

Proof. For simplicity we only show this when $d = 2$ as the proof is essentially the same for any other d. We start by showing that the $\mathcal{T}_{\mathbb{C}^2}$ has elements satisfying the relations of $\mathcal{E}_2$. Indeed, consider $v_1 := c_{(1,0)}$ and $v_2 := c_{(0,1)}$. We have to check that $v_1^* v_1 = v_2^* v_2 = 1$ in $\mathcal{L}_{\mathbb{C}}(\mathcal{F}(\mathbb{C}^2))$. We only do $v_1^* v_1 = 1$, the other one being analogous. We have to check $v_1^* v_1$ acts as the identity on each $(\mathbb{C}^2)^{\otimes n}$. Well, for $n = 0$ take $z \in \mathbb{C}$

$$(v_1^* v_1)(z) = c_{(1,0)}^*(z, 0) = \langle (1, 0), (z, 0)\rangle = z.$$

For $n \geq 1$,

$$(v_1^* v_1) = c_{(1,0)}^* c_{(1,0)} = \varphi(\langle (1, 0), (1, 0)\rangle) = \varphi(1) = \text{id}$$

By universality there is a unique $*$-homomorphism $\psi : \mathcal{E}_2 \to \mathcal{T}_{\mathbb{C}^2}$, which sends the s_j to v_j. Notice that ψ is surjective because v_1, v_2 generate $\mathcal{T}_{\mathbb{C}^2}$. To show that it's injective, we use the universal property of $\mathcal{T}_{\mathbb{C}^2}$ we produce a map $\mathcal{T}_{\mathbb{C}^2} \to \mathcal{E}_2$ which is a left inverse to ψ. Let $\pi : \mathbb{C} \to \mathcal{E}_2$ be given by $\pi(z) := z1$ and $t : \mathbb{C}^2 \to \mathcal{E}_2$ be given by

$$t(\zeta_1, \zeta_2) = \zeta_1 s_1 + \zeta_2 s_2$$

It's obvious that π is a $*$-homomorphism and that t is a linear map. Moreover, using that $s_j^* s_j = 1$ we get

$$t(\zeta_1, \zeta_2)^* t(\eta_1, \eta_2) = (\overline{\zeta_1}\eta_1 + \overline{\zeta_2}\eta_2)1 = \pi(\langle (\zeta_1, \zeta_2), (\eta_1, \eta_2)\rangle)$$

Finally,

$$\pi(z)t(\zeta_1,\zeta_2) = z1(\zeta_1 s_1 + \zeta_2 s_2) = z\zeta_1 s_1 + z\zeta_2 s_2 = t(\varphi(z)(\zeta_1,\zeta_2))$$

Hence, universality gives the $*$-homomorphism $\widehat{\pi} : \mathcal{T}_{\mathbb{C}^2} \to \mathcal{E}_2$ sending c_ξ to $t(\xi)$. Since $t(1,0) = s_1$ and $t(0,1) = s_2$, it follows $\widehat{\pi}$ is indeed a left inverse for ψ. ■

Remark 4.4. Notice that $\mathcal{T}_{\mathbb{C}^2} \neq \mathcal{O}_2$. Indeed, if $v_1 := c_{(1,0)}$ and $v_2 := c_{(0,1)}$ are asin the proof, then $v_1v_1^* + v_2v_2^*$ does not quite act as the identity on all degrees of $\mathcal{F}(\mathbb{C}^2)$. In fact, it only fails to do so at degree 0, because the adjoint kills everything at $n = 0$. Indeed,

$$(v_1v_1^* + v_2v_2^*)\big((\kappa_n)_{n\geq 0}\big) = \big((\kappa_n)_{n\geq 1}\big)$$

Thus, $1 - (v_1v_1^* + v_2v_2^*)$ is in fact a rank one operator who extracts the 0 coefficient, that is

$$1 - (v_1v_1^* + v_2v_2^*) = \theta_{(1,0,0,\ldots),(1,0,0,\ldots)}$$

This problem will no longer occur when we define the Cuntz-Pimsner algebra $\mathcal{O}_{\mathbb{C}^2}$, as we will be taking a quotient by a set that contains $1 - (v_1v_1^* + v_2v_2^*)$.

A more general version of Theorem 4.3 above, with essentially the same proof, is the following

Theorem 4.5. *Let $d \in \mathbb{Z}_{\geq 2}$,$A$ a C^*-algebra and regard A^d as a Hilbert A-module. Let $\varphi : A \to \mathcal{L}_A(A^d)$ by given by*

$$\varphi(a)(a_1,\ldots,a_d) := (aa_1,\ldots,aa_d)$$

Then (A^d,φ) is a C^* correspondence and $\mathcal{T}_{A^d} \cong A \otimes \mathcal{E}_d$.

5 The Cuntz-Pimsner algebra $\mathcal{O}_E$ and $\mathcal{O}_{\mathbb{C}^d} \cong \mathcal{O}_d$

Definition 5.1. For a C* correspondence (E,φ) over A, we define the ideal $J_E := \varphi^{-1}(\mathcal{K}_A(E))$.

Lemma 5.2. *Let (E,φ) be a C^* correspondence over A. Then $\mathcal{F}(E)J_E$ is a Hilbert J_E-module and*

$$\mathcal{K}_{J_E}(\mathcal{F}(E)J_E) = \overline{\operatorname{span}}\{\theta_{\kappa a,\tau} : \kappa,\tau \in \mathcal{F}(E), a \in J_E\} \trianglelefteq \mathcal{L}_A(\mathcal{F}(E))$$

We will now consider the quotient C*-algebra $\mathcal{Q}_A(E) := \mathcal{L}_A(\mathcal{F}(E))/\mathcal{K}_{J_E}(\mathcal{F}(E)J_E)$ together with the quotient map $q : \mathcal{L}_A(\mathcal{F}(E)) \to \mathcal{Q}_A(E)$.

Definition 5.3. Let (E,φ) be a C^*-correspondence over A. We define $\mathcal{O}_E$, the Cuntz-Pimsner algebra of E, as the C^*-subalgebra in $\mathcal{Q}_A(E)$ generated by the image of the creation operators $\{q(c_\xi) : \xi \in E\}$.

Remark 5.4. Just as before, we assume that E is a full Hilbert A-module and identify A as a subset of $\mathcal{O}_E$ via $a \mapsto q(\varphi(a))$.

The Cuntz-Pimsner algebra $\mathcal{O}_E$ has the following universal property. Suppose B is another C^*-algebra and that there is a $*$-homomoprhism $\pi : A \to B$, a linear map $t : E \to B$ such that

1. $t(\xi)^*t(\zeta) = \pi(\langle\xi,\zeta\rangle)$ for $\xi,\zeta \in E$,
2. $\pi(a)t(\xi) = t(\varphi(a)\xi)$

Further, consider the $*$-homomorphism $\Theta_t : \mathcal{K}_A(E) \to B$ satisfying $\Theta_t(\theta_{\xi,\zeta}) = t(\xi)t(\zeta)^*$ for $\xi,\zeta \in E$, and suppose also that

3. $\pi(a) = \Theta_t(\varphi(a))$ for $a \in J_E$

Then there π has is a unique extension $\widehat{\pi} : \mathcal{O}_E \to B$ that sends $q(c_\xi)$ to $t(\xi)$.

Theorem 5.5. *Let $d \in \mathbb{Z}_{\geq 2}$ and regard $\mathbb{C}^d$ as a Hilbert $\mathbb{C}$-module. Let $\varphi : \mathbb{C} \to \mathcal{L}_{\mathbb{C}}(\mathbb{C}^d)$ by given by*

$$\varphi(z)(\zeta_1, \dots, \zeta_d) := (z\zeta_1, \dots, z\zeta_d)$$

Then $(\mathbb{C}^d, \varphi)$ is a C^* correspondence and $\mathcal{O}_{\mathbb{C}^d} \cong \mathcal{O}_d$.

Proof. Again we only show this when $d = 2$. We will show that $\mathcal{O}_{\mathbb{C}^2}$ fits the universal property for $\mathcal{O}_2$. Well, if $v_1 := q(c_{(1,0)})$ and $v_2 := q(c_{(0,1)})$. Using our computations from Theorem 4.3 we have

$$v_1^* v_1 = v_2^* v_2 = q(\mathrm{id}_{\mathcal{L}_A(\mathcal{F}(\mathbb{C}^2))}) = 1$$

Also, notice that $\theta_{(1,0,0,\dots),(1,0,0,\dots)} \in \mathcal{K}_{J_E}(\mathcal{F}(E)J_E)$ because in this case $\mathcal{L}_C(\mathbb{C}^2) = \mathcal{K}_{\mathbb{C}}(\mathbb{C}^2)$ and therefore $J_{\mathbb{C}^2} = \mathbb{C}$. Thus, following the computations from Remark 4.4 we have

$$v_1 v_1^* + v_2 v_2^* = q(\mathrm{id} - \theta_{(1,0,0,\dots),(1,0,0,\dots)}) = q(\mathrm{id}) = 1$$

Hence, universality gives a surjective $*$-homomorphism $\mathcal{O}_2 \to \mathcal{O}_{\mathbb{C}^2}$ sending s_j to v_j, for $j = 1, 2$. Since $\mathcal{O}_2$ is simple,such homomorphism has to be injective and we are done. ■

A more general version of Theorem 5.5 above, with essentially the same proof, is the following

Theorem 5.6. *Let $d \in \mathbb{Z}_{\geq 2}$,$A$ a C^*-algebra and regard A^d as a Hilbert A-module. Let $\varphi : A \to \mathcal{L}_A(A^d)$ by given by*

$$\varphi(a)(a_1, \dots, a_d) := (aa_1, \dots, aa_d)$$

Then (A^d, φ) is a C^* correspondence and $\mathcal{O}_{A^d} \cong A \otimes \mathcal{O}_d$.

6 A potential L^p version

If (X, μ) is a measure space and $p \in [1, \infty)$, an L^p-operator algebra A is a Banach algebra that's isometrically isomorphic to a norm closed subalgebra of $\mathcal{L}(L^p(X, \mu))$.

Definition 6.1. Let $d \geq 2$ be an integer. We define the **Leavitt algebra** L_d to be the universal complex unital algebra generated by elements $s_1, s_2, \dots, s_d, t_1, t_2, \dots, t_d$ satisfying

$$t_j s_k = \delta_{j,k} \quad \text{and} \quad \sum_{j=1}^{d} s_j t_j = 1$$

There is a well defined norm on L_d that comes from a particular kind of algebraic representations of L_d on σ-finite L^p spaces. The completion of L_d with respect to this norm is the L^p-Cuntz algebra $\mathcal{O}_d^p$.

I have been working on a Fock space-type construction for an L^p operator algebra A that yields a class of L^p-operator algebras that contains the L^p-Cuntz algebras. Looks like looking at the notion of Rigged Hilbert modules introduced by D.P. Blecher gives a reasonable starting point. One needs to give an operator space type definition for L^p spaces. Some issues that might appear along the road will require to fix a representation $\pi : A \to \mathcal{L}(L^p(X, \mu))$, choose a particular tensor product and a particular type of completion.

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